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J. Michael Steele

Stochastic Calculus and Financial Applications

Springer

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Preface

This book is designed for students who want to develop professional skill in stochastic calculus and its application to problems in finance. The Wharton School course that forms the basis for this book is designed for energetic students who have had some experience with probability and statistics but have not had advanced courses in stochastic processes. Although the course assumes only a modest background, it moves quickly, and in the end, students can expect to have tools that are deep enough and rich enough to be relied on throughout their professional careers.

The course begins with simple random walk and the analysis of gambling games. This material is used to motivate the theory of martingales, and, after reaching a decent level of confidence with discrete processes, the course takes up the more demanding development of continuous-time stochastic processes, especially Brownian motion. The construction of Brownian motion is given in detail, and enough material on the subtle nature of Brownian paths is developed for the student to evolve a good sense of when intuition can be trusted and when it cannot. The course then takes up the Itô integral in earnest. The development of stochastic integration aims to be careful and complete without being pedantic.

With the Itô integral in hand, the course focuses more on models. Stochastic processes of importance in finance and economics are developed in concert with the tools of stochastic calculus that are needed to solve problems of practical importance. The financial notion of replication is developed, and the Black-Scholes PDE is derived by three different methods. The course then introduces enough of the theory of the diffusion equation to be able to solve the Black–Scholes partial differential equation and prove the uniqueness of the solution. The foundations for the martingale theory of arbitrage pricing are then prefaced by a well-motivated development of the martingale representation theorems and Girsanov theory. Arbitrage pricing is then revisited, and the notions of admissibility and completeness are developed in order to give a clear and professional view of the fundamental formula for the pricing of contingent claims.

This is a text with an attitude, and it is designed to reflect, wherever possible and appropriate, a prejudice for the concrete over the abstract. Given good general skill, many people can penetrate most deeply into a mathematical theory by focusing their energy on the mastery of well-chosen examples. This does not deny that good abstractions are at the heart of all mathematical subjects. Certainly, stochastic calculus has no shortage of important abstractions that have stood the test of time. These abstractions are to be cherished and nurtured. Still, as a matter of principle, each abstraction that entered the text had to clear a high hurdle.

Many people have had the experience of learning a subject in 'spirals.' After penetrating a topic to some depth, one makes a brief retreat and revisits earlier topics with the benefit of fresh insights. This text builds on the spiral model in several ways. For example, there is no shyness about exploring a special case before discussing a general result. There also are some problems that are solved in several different ways, each way illustrating the strength or weakness of a new technique.

Any text must be more formal than a lecture, but here the lecture style is followed as much as possible. There is also more concern with 'pedagogic' issues than is common in advanced texts, and the text aims for a coaching voice. In particular, readers are encouraged to use ideas such as George Pólya's "Looking Back" technique, numerical calculation to build intuition, and the art of guessing before proving. The main goal of the text is to provide a professional view of a body of knowledge, but along the way there are even more valuable skills one can learn, such as general problem-solving skills and general approaches to the invention of new problems.

This book is not designed for experts in probability theory, but there are a few spots where experts will find something new. Changes of substance are far fewer than the changes in style, but some points that might catch the expert eye are the explicit use of wavelets in the construction of Brownian motion, the use of linear algebra (and dyads) in the development of Skorohod's Embedding, the use of martingales to achieve the approximation steps needed to define the Itô integral, and a few more.

Many people have helped with the development of this text, and it certainly would have gone unwritten except for the interest and energy of more than eight years of Wharton Ph.D. students. My fear of omissions prevents me from trying to list all the students who have gone out of their way to help with this project. My appreciation for their years of involvement knows no bounds.

Of the colleagues who have helped personally in one way or another with my education in the matters of this text, I am pleased to thank Erhan Çinlar, Kai Lai Chung, Darrell Duffie, David Freedman, J. Michael Harrison, Michael Phelan, Yannis Karatzas, Wenbo Li, Andy Lo, Larry Shepp, Steve Shreve, and John Walsh. I especially thank Jim Pitman and Ruth Williams for their comments on an early draft of this text. They saved me from some grave errors, and they could save me from more if time permitted. Finally, I would like to thank Vladimir Pozdnyakov for hundreds of hours of conversation on this material. His suggestions were especially influential on the last five chapters.

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